

$c > 1$ Non-Critical Strings and Large- N Matrix Field Theory

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Abstract

Motivated by a possible relativistic string description of hadrons we use a discretised light-cone quantisation and Lanczos algorithm to investigate the phase structure of ϕ^3 matrix field theory in the large N limit. In $1 + 1$ dimensions we confirm the existence of Polyakov's non-critical string theory at the boundary between parton-like and string-like phases, finding critical exponents for longitudinal oscillations equal to or consistent with those given by a mean field argument. The excitation spectrum is finite, possibly discrete. We calculate light-cone structure functions and find evidence that the probability $Q(x)$ of a parton in the string carrying longitudinal momentum fraction between x and $x + dx$ has support on all $0 < x < 1$, despite the average number of partons being infinite.

1 Light-Cone Strings.

Soon after the seminal work of Goddard et al. [1] on canonical light-cone quantisation of relativistic string theory, it was realised that the restriction to critical central charge c of matter might be avoided if motions of the string were allowed which made it impossible to go to light-cone gauge. Point-like insertions of energy-momentum [2, 3] result in a singular space-time co-ordinate function $X(\sigma)$ so that a smooth change of variable $\sigma \rightarrow \sigma'$ on the string is unable to redistribute the longitudinal momentum P^+ evenly along the string. In this way the longitudinal oscillations of the string remain as physical degrees of freedom. However the longitudinal oscillations are in general non-linear and the difficulty in studying such systems has left a dearth of reliable results for non-critical strings in physical spacetime dimensions. These theories are interesting as models for hadrons, this being the principal motivation behind the work of Green [3] for example. Point-like insertions of energy momentum on the string give it a character intermediate between that of critical strings and a finite collection of partons, and therefore there is a possibility to describe both the parton-like and string-like behaviour of hadrons in a single transparent framework. Matrix field theories [4, 5] offer an explicit implementation of this idea. In this paper we continue the study of the discretised light-cone quantisation (DLCQ) of matrix models with $c \geq 2$ introduced in ref.[6] and further studied numerically [7] and analytically [8], by applying a Lanczos algorithm to the mass-matrix evaluation. We find a parton-like phase, and also string-like phase in which longitudinal dynamics are essentially frozen, corresponding to the string of Goddard et al. at non-critical c [9, 10]. These two phases are separated by a transition at which there exists a non-critical string theory, essentially that of Polyakov [11], with non-trivial longitudinal degrees of freedom. Although a stringy object it seems to exhibit parton-like behaviour in structure functions.

We consider an $N \times N$ hermitian matrix field $\phi_{ab}(x)$ in d dimensions subject to the following (Euclidean) action

$$S = \int d^d x \text{Tr} \left(\frac{1}{2} (\partial_\alpha \phi)^2 + \frac{1}{2} \mu \phi^2 - \frac{1}{3\sqrt{N}} \lambda \phi^3 \right) . \quad (1)$$

It is well-known that the Feynman diagram $1/N$ expansion, when considered in terms of the dynamical triangulations given by the dual graphs [5, 12], represents an explicit discretisation of the Polyakov path integral [11] for random surfaces, albeit with an exponential rather than Gaussian propagator. It may be regarded as a prototype QCD in the sense that it is a theory of fields transforming in the adjoint representation of (in this case global) $SU(N)$, from which one can build a flux-tube/string representation of the bound states. In the context of light-cone quantisation the relationship is more precisely stated in ref.[13] as a dimensional reduction. The observables we consider are the closed strings of n partons at some fixed time (prototype glueballs)

$\text{Tr}[\phi(x_1) \dots \phi(x_n)]$ and products of these (multi-string states). If mass eigenstates have $\langle n \rangle$ finite they will be called parton-like, while if $\langle n \rangle = \infty$ they will be referred to as string-like. Applying light-cone quantisation in Minkowski space, one derives the spectrum as a function of the bare worldsheet cosmological constant $\log \lambda$ and can search for critical behaviour in this parameter. Such behaviour can occur since, according to general arguments [14], the sum of graphs at a given order in $1/N$ in a UV finite theory should itself be finite for sufficiently small coupling constant λ ; the $1/N$ expansion on the other hand is only asymptotic as a result of the unboundedness of the action (1). As $\lambda \rightarrow \lambda_c$ one approaches the edge of the domain of convergence, at which large graphs are favoured, which in light-cone formalism means a transition from a parton-like to a string-like phase. We shall consider the $N \rightarrow \infty$ limit and study the UV-finite normal-ordered $1+1$ -dimensional theory, commenting at the end on the more general case.

Let us now rotate $x^0 \rightarrow ix^0$ and find the relativistic spectrum by light-cone quantisation, treating $x^+ = (x^0 + x^1)/\sqrt{2}$ as time and $k^+ = (k^0 + k^1)/\sqrt{2} > 0$ as (longitudinal) momentum. This has been described before [6] and so we make only brief comments. The problem can be expressed in terms of the Fourier modes $a_{ij}(k^+)$ of ϕ_{ij} . We will discretize the problem to render the number of states of total momentum P^+ finite by allowing partons with $k^+ = mP^+/K$ where m is a positive integer and $K \geq m$ is a fixed large integer which plays the role of cutoff [9, 15]. Then in the quantum theory

$$\delta_{mm'}\delta_{il}\delta_{jk} = [a_{ij}(m), a_{lk}^\dagger(m')] , \quad (2)$$

and

$$2P^+P^- = \mu K(V - yT) , \quad y = \frac{\lambda}{2\mu\sqrt{\pi}} , \quad (3)$$

where

$$V = \sum_{m=1}^K \frac{1}{m} a_{ij}^\dagger(m) a_{ij}(m) , \quad (4)$$

$$T = \frac{1}{\sqrt{N}} \sum_{m_1, m_2=1}^K \frac{a_{ij}^\dagger(m_1+m_2) a_{ik}(m_2) a_{kj}(m_1) + a_{ik}^\dagger(m_1) a_{kj}^\dagger(m_2) a_{ij}(m_1+m_2)}{\sqrt{m_1 m_2 (m_1+m_2)}} . \quad (5)$$

Single closed-string states are in general linear combinations of the Fock states

$$\Psi = \sum_{n=1}^{\infty} \sum_{m_i=1}^K \delta(K - \sum_{i=1}^n m_i) f_n(m_1, \dots, m_n) N^{-n/2} \text{Tr}[a^\dagger(m_1) \dots a^\dagger(m_n)] |0\rangle . \quad (6)$$

The light-cone energy P^- satisfies : $P^- : |0\rangle = 0$. The spectrum $M^2 = 2P^+P^-$ can be found by diagonalising the matrix eigenvalue problem (3) in the basis of discretised Fock states, labelled by the ordered partitions of K .¹ To establish the basic stringy scaling laws we will study the $N \rightarrow \infty$

¹A simple example at $K = 3$ can be found in ref.[6].

free-string limit (not equivalent to free partons!). There are many fewer states to consider in this case, since $1/N$ is the string coupling and only single string states need be retained, leading to a linear light-cone Schrodinger equation for Ψ [9, 10]. The eigenfunctions give directly the structure functions of the string in terms of the longitudinal momentum fractions $x = m/K$. The light-cone quantisation we performed is naive in the sense that we have neglected zero modes $a^\dagger(0)$. This was done partly for simplicity but also to regulate the theory in the string phase as will become clear shortly.

2 Numerical Results

The number of Fock states increases exponentially with K and the exact determination (and diagonalisation) of the matrix representation of M^2 soon becomes onerous, e.g. at $K = 13, 14, 15, 16$ there are 631, 1181, 2191, 4115 states respectively. Nevertheless, if we wish to consider only the extreme eigenstates of an operator, such as the ground state and near ground states, then a considerable amount of computer processing time can be saved by implementing numerical approximation schemes that do not require a precise knowledge of the operator's matrix representation. We used the Lanczos algorithm, in the context of DLCQ specific applications of which are detailed in ref.[16]. In the context of our own work, all the necessary numerical and symbolic manipulations were performed with the help of Mathematica and Sun workstations. The Lanczos algorithm is iterative by nature, and a judicious choice of initial state is necessary to ensure proper convergence. It was soon discovered that any straightforward application of this method involves the manipulation of a rapidly growing number of states at each subsequent step in the iteration. This arises because successive application of P^- on a typical Fock state yields a proliferation of new states that grows rapidly as the iteration proceeds, and it is this property that severely limits the scope of any iterative approximation scheme. Progress can be made, however, by recognizing that the action of the hamiltonian on existing Fock states is repeated many times during the iteration process, so unnecessary computations can be avoided by 'remembering' where a single string Fock state ends up after it has first been acted on by the light-cone hamiltonian. In computing terminology, this amounts to introducing 'pointers' between Fock states during the course of the programme. Implementation of this scheme enabled $K \leq 16$ to be studied. The one parton state $N^{-1/2}\text{Tr}[a^\dagger(K)]|0\rangle$ seems an adequate initial state for the Lanczos algorithm, and convergence was assessed to at least 3 s.f. accuracy in anything we plot in this paper.

In ref.[8] a mean field theory was used to map out the structure of the lowest mass eigenstate of the $1+1$ -dimensional theory (1) at $K = \infty$ as $y = \lambda/2\mu\sqrt{\pi}$ is increased. At $y = 0$ this is

the one-parton state. For $y < y_c \approx 0.53$ states with a few partons dominate in the wavefunction, while at $y > y_c$ each parton typically carries a finite amount of discretised momentum m , which decreases with increasing y , and the wavefunction typically carries an infinite number of partons. In this latter phase the longitudinal dynamics are trivial, essentially those of a theory with all $m = 1$ ($n = K$) [9, 10]. This picture is only valid when zero modes are neglected – for a given K this regulates the instability to creation of more and more partons when $y > y_c$. At $y = y_c$ on the other hand there are typically an infinite number of partons each carrying an infinite amount of discrete momentum m , so that the longitudinal dynamics are non-trivial. At $y > y_c$ mean field predicted the scaling laws $\langle n \rangle \propto K$ and $M^2 \propto -K^2$ as $K \rightarrow \infty$, laws which appear to be exact, and so the quantities M^2/K^2 and $\langle n \rangle / K$ can be used as a signal for the string phase. We plot them in Figs. 1(a) and 1(b), both for the mean field² ($K = \infty$) and an exact diagonalisation of the matrix M^2 at $K = 12$ (351 states). In fact mean field predicts that as $y - y_c \rightarrow 0^+$, $M^2/K^2 \propto -(y - y_c)^2$ and $\langle n \rangle / K \propto (y - y_c)$.

The probability $P(n)$ of finding n partons and the probability density $Q(x)$ of finding a parton with momentum fraction x in the groundstate gives information about the internal structure of the string in each phase. We computed these functions for $y < y_c$, $y \approx y_c$, and $y > y_c$. At $y < y_c$ we typically found that $P(n)$ and $Q(x)$ were essentially delta-functions at $n = 1$ and $x = 1$, indicating that very little mixing into higher Fock states occurs. In Figs. 2(a) and 2(b) we plot them for $y = 0.53$ and $K = 12$. The length distribution of strings $P(n)$ is still peaked at $n = 1$, a peak which falls with increasing K , but now has contributions from all other lengths. Similarly the structure function $Q(x)$ is still peaked at $x = 1$ but with a shallow secondary peak at $x \sim 0.5$, the latter presumably a manifestation of the dominant $1 \rightarrow 2$ decay mode of the one parton state — mean field predicts that $\langle n \rangle \sim 2$ at $K = 12$ and $y = y_c$. For $y > y_c$ the behaviour is quite different, shown in also at $K = 12$ in Figs. 2(a) and 2(b) for $y = 1$. Now $P(n)$ is peaked at $n \propto K$ with increasing K , while $Q(x)$ is peaked at low x already at $K = 12$. This indicates that the partons carry a finite amount of discrete momentum in this phase, while at the transition point this is was not necessarily true. Figure 3 shows how $Q(0.5)$ at $y = 0.53$ changes with K . Remarkably it seems to tend to a non-zero value as $K \rightarrow \infty$, indicating that although $\langle n \rangle = \infty$ and we are dealing with a string theory, the structure function has finite support on the whole interval $(0, 1]$. We checked also that at $y > y_c$, $Q(0.5)$ falls with increasing K and is consistent with zero at $K = \infty$.

To estimate the critical exponents $\langle n \rangle \propto K^\gamma$ and $M^2 \propto K^\delta$ we used log-log plots with

²There is a scale ambiguity in the mean field coupling y which we have fixed from the position of the true y_c estimated from the exact diagonalisations.

increasing K . For $y > 0.53$ we found rapid convergence already at small K , showing with certainty that the mean field exponents are exact: $\gamma = 1$; $\delta = 2$. Similarly rapid convergence both as a function of K and number of iterations of the Lanczos algorithm is found in the parton phase $y < y_c$. For $y \approx y_c$ the large fluctuations characteristic of a critical point make convergence much slower. We expect the forms

$$\frac{\langle n \rangle}{K} = a_1(y - y_c)^a + a_2 K^{\gamma-1} + \dots \quad (7)$$

$$\frac{M^2}{K^2} = b_1(y - y_c)^b + b_2 K^{\delta-2} + \dots \quad (8)$$

where the ellipses indicate higher orders in $1/K$. Mean field predicts $a = 1$ and $b = 2$, which we assume are not far from the true values (see fig.1). If we also assume that the coefficients a_i and b_i are not too large or small then γ and δ can be estimated provided we get close enough to the critical point and have high enough K . Figure 4 plots $\log[\langle n \rangle / \sqrt{K}]$ at $y = 0.53$. We are confident that the true y_c is within 0.01 so that with the above assumptions finite K is really the only limitation since we typically measured $\langle n \rangle / K \sim 0.1$. There is a degree of uncertainty since we have not clearly reached a straight-line scaling region, which would be horizontal if the mean field exponent were exact, but taking the slope at the highest K gives an upper bound at $y = y_c$ of $\gamma < 0.6$ (meanfield 0.5). At $K \sim 16$ we find that $M^2/K^2 \sim -0.0001$ at $y = 0.53$, which is still probably of the same order as the leading term in eq.(8). Therefore we were unable to estimate δ in this case (mean field value $\delta = 1$).

For $y > y_c$ we have found similar scaling laws hold for excited states, whose squared masses are separated by $O(K^2)$ gaps. Thus the longitudinal *fluctuations* with respect to the groundstate are indeed trivial in this phase. The behaviour of excited states with respect to the groundstate was also studied as a function of y in refs.[6, 7], with the result that the gaps in the spectrum appeared finite for $y < y_c$. To study this question as $y - y_c \rightarrow 0^-$ we calculated the mass difference of ground and first excited state at $y = 0.53$ for increasing K , shown in Figure 5. It clearly favours a finite spectrum. Unfortunately the data cannot decide for sure between continuous or discrete spectrum, but the latter is not ruled out.

3 Discussion

In the limited space left to us here let us address a couple of the questions posed by the preceding analysis. We have analysed the $1 + 1$ dimensional theory explicitly here, but in principle one could add some transverse degrees of freedom. In order to assess the effect of these we studied the simplest case, that of Ising degrees of freedom on the string, i.e. a transverse lattice consisting of

two points. This can be solved in a mean field approximation [17] by an analysis similar to that in ref.[8], and we found that the exponent γ was unchanged for all values of the Ising model coupling constant. If the mean field is trustworthy, it leads one to believe that transverse degrees of freedom do not change the scaling laws for the longitudinal modes. It is of course quite possible that the excitation spectrum acquires additional states at finite energy with respect to the groundstate for some critical Ising coupling, a fact which can be proved explicitly [17] in the $y \rightarrow \infty$ limit along the lines of refs.[9, 10, 18].

Although the $1/N$ expansion of the field theory (1) has no pathologies as such, we have taken various steps to isolate a string theory, such as neglecting non-singlets and zero modes, which questions the self-consistency. A full discussion of consistency is beyond the scope of this letter but we mention the problem of renormalisation of the mass divergence. Presumably this can be cured by some procedure analogous to those employed for critical strings, where there is a similar divergence (of opposite sign however). For example, mean field predicts that the groundstate mass squared diverges like $\sim -K$ at $y = y_c$ which, if correct, could be subtracted by adding a constant to the Lagrangian density (1). Further consistency conditions may serve to fix the finite part, but we see no precise reason at this stage why it should be tachyonic.

Our results have suggested that the free non-critical bosonic string coupled to a point source should exhibit a parton-like structure function, and we could not rule out discrete non-tachyonic spectrum. Moreover the work of Green [3] indicated that high energy fixed angle scattering of strings similar to ours shows power law behaviour. All these properties are of course those of actual hadrons. Obviously our numerical results must be investigated further to check that they are not finite cutoff artifacts, but there seems no obstacle to more extensive numerical calculations using the Lanczos algorithm we have implemented here, and it seems worthwhile to pursue with renewed vigour analytic approaches to $c > 1$ non-critical string theory.

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FIGURE CAPTIONS

Fig.1. – (a) M^2/K^2 for the groundstate ($\mu = 1$). (b) $\langle n \rangle / K$ in the groundstate. Broken line is mean field result at $K = \infty$, solid line is exact result at $K = 12$.

Fig.2. – (a) Probability $P(n)$ of finding n partons in the groundstate. (b) Probability $Q(x)$ of finding a parton with longitudinal momentum fraction between x and $x + dx$ in the groundstate. Solid circles are $y = 0.53$, open circles are $y = 1$.

Fig.3. – $Q(0.5)$ for increasing cut-off K at $y = 0.53$

Fig.4. – $\log \langle n \rangle / \sqrt{K}$ for increasing K at $y = 0.53$.

Fig.5. – $\Delta M^2 = M_1^2 - M_0^2$, the difference in mass squared ($\mu = 1$) between ground and first excited state, for increasing K at $y = 0.53$.

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